Mathematics of Rivalry: Modelling War, Peace and Competition

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Goal & Overview

- Introduce concept and *mathematics of competitive situations*: Lanchester approach, Lotka-Volterra competition, Wright/Kauffman fitness landscapes
- Competition: Use tools from ODEs/PDEs and complex adaptive systems theory to describe, model and interpret dynamics (Lanchester, Lotka-Volterra, Kolmogorov)
- Negotiation: Pick pricing state space ("fitness landscapes"), study the system's macro topology and use geometric insight to understand pathways and dynamics (Wright, Kauffman)
- Application: Construct and interpret pricing landscape ("pricescape") for major new antibiotics launched since 2002

Mathematics of War

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Historical Starting Point I: Fighting Strength, Lanchester's Equations 1917

- A simple battle model: suppose that $R(t)$ red and $G(t)$ green units begin fighting at $t = 0$, and that each unit destroys r or g (the fighting effectiveness) enemy units in one time unit, s.t.

$$\frac{dR}{dt} = -gG, \quad \frac{dG}{dt} = -rR$$

- To solve eliminate the explicit t-dependence by dividing the second equation by the first and then by separating variables

$$\begin{aligned} \int rRdR &= \int gGdG \\ rR^2 - gG^2 &= \text{ constant} \end{aligned}$$

- $rR^2 - gG^2$ is constant, only one of R or G approaches zero (only one wins). The fighting strengths (rR^2, gG^2) *per se* vary by fighting effectiveness times the square of their numbers.

Two Laws: Lanchester's Aimed vs. Unaimed Fire

- Lanchester's Square Law (*Aimed* Fire): The ODE system

$$\dot{x} = -\beta y; x > 0$$
$$\dot{y} = -\alpha x; y > 0$$

represents a situation where attrition to each side is proportional to the number of units remaining on the other, and there are no reinforcements. Its solution is $\alpha\left(y^2 - y_0^2\right) = \beta\left(x^2 - x_0^2\right)$, hence the name *square* law.

- Lanchester's Linear Law (*Unaimed* Fire): The ODE system

$$\dot{x} = -\beta xy; x > 0$$
$$\dot{y} = -\alpha xy; y > 0$$

means x's fire is merely directed into v's operating area, rather than being aimed at a specific v unit, then the attrition rate for v will be proportional to y, as well as to x. Its solution is $\alpha\left(x - x_0\right) = \beta\left(y - y_0\right)$, hence the name *linear* law.

Fighting Strength: Bracken's Generalized Model

- Attempts in the literature to fit either the aimed- or the unaimed-fire model to data have only been partly successful. One can use Bracken's generalized model whose parameters one then fits to the data:

$$\frac{dR}{dt} = -gR^q G^p, \quad \frac{dG}{dt} = -rR^p G^q$$

for p and q to be empirically determined. The conserved quantity (by eliminating t, separating and integrating) is

$$gG^\alpha - rR^\alpha$$

where $\alpha = 1 + p - q$ (the Lanchester aimed-fire model corresponds to $p = 1, q = 0$ and thus to $\alpha = 2$, the unaimed-fire model to $p = q = 1$ and thus $\alpha = 0$).

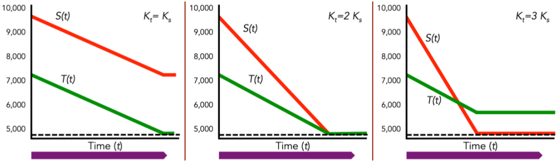
Summary: Lanchester Type Differential Equations

- Functional Forms for Lanchester Combat Models

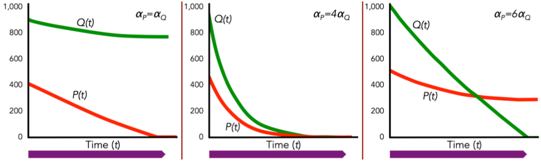
Attrition Process	Differential Equation	State Equation
F/F	$\frac{dx}{dt} = -\alpha y$ $\frac{dy}{dt} = -\beta x$	$\beta(x_0^2 - x^2) = \alpha(y_0^2 - y^2)$ Square Law
FT/FT	$\frac{dx}{dt} = -\alpha xy$ $\frac{dy}{dt} = -\beta xy$	$\beta(x_0 - x) = \alpha(y_0 - y)$ Linear Law
F/FT	$\frac{dx}{dt} = -\alpha y$ $\frac{dy}{dt} = -\beta xy$	$\frac{\beta}{2}(x_0^2 - x^2) = \alpha(y_0 - y)$ Brackney's Mixed Law

Solutions to Lanchester Equations

- Lanchester's Linear Law (unaimed fire, one-on-one combat): (fighting strength K_i)



- Lanchester's Square Law (aimed fire, ranged combat): (attrition rate α_i)



Mathematics of Competition

Historical Starting Point II: Predator & Prey, Lotka-Volterra Equations 1926

- Lotka-Volterra models are nonlinear, mixed Lanchester models
- Competition equations add another term to account for limitations of the growth rate imposed by members of other populations.
- The corresponding simplified Lotka-Volterra equations are

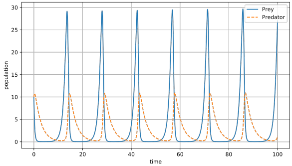
$$\frac{dN_1}{dt} = \alpha N_1 - \beta N_1 N_2, \frac{dN_2}{dt} \quad = \delta N_1 N_2 - \gamma N_2$$

where

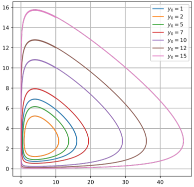
N_1 is the number of prey; N_2 is the number of predators;
$\frac{dN_1}{dt}$ and $\frac{dN_2}{dt}$ are growth rates of the two populations;
t represents time;
$\alpha, \beta, \gamma, \delta (> 0)$ describe the interaction of the two species.

Predator & Prey: Solutions to Lotka-Volterra Equations

- The equations have periodic solutions and do not have a simple expression in terms of the usual trigonometric functions

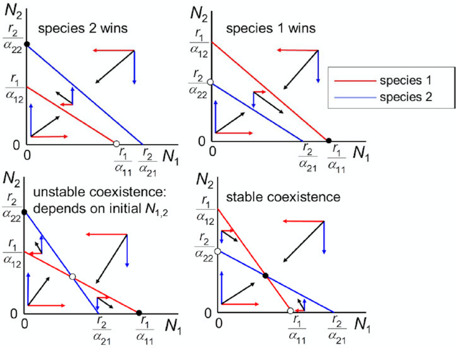


- Solutions presented as orbits in phase space (eliminating time); one axis: n_{prey}, other axis: $n_{predators}$



Predator & Prey: Equilibrium, Lotka-Volterra Isoclines

- Gause-Witt analysis, two-species Lotka-Volterra competition



- The isocline for each species i is the line on the N_1/N_2 phase plane where $dN_i/dt = 0$. Joint equilibria shown as filled (stable) and hollow (unstable) circles.

Mathematics of Negotiation

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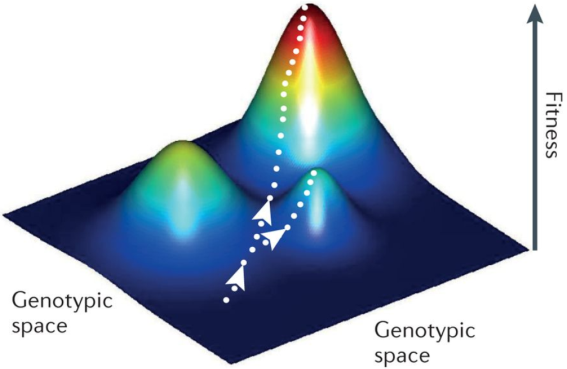
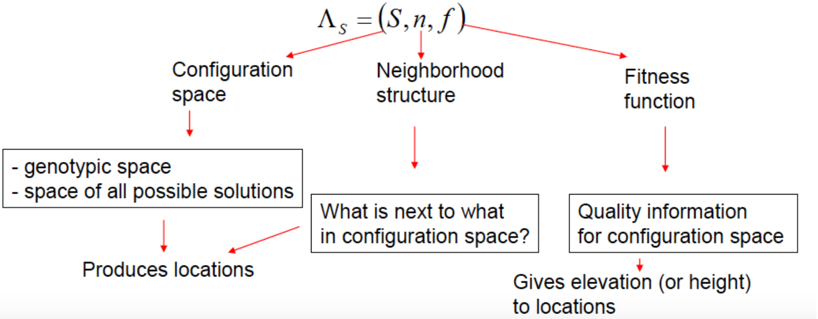
Historical Starting Point III: Fitness Landscapes in Biology, Wright 1932

- Fitness landscapes in biology: visualize/measure relationships btw. genotypes and reproductive success



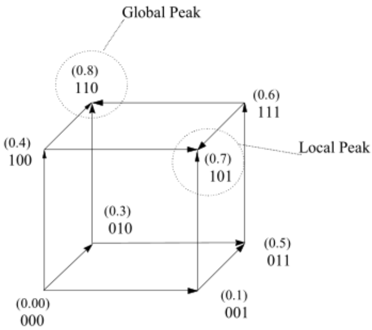
Fitness Landscapes: Formalization, Interpretation



Kauffman's NKAC Model: Genetics *vs.* Pricing Economics

Term	Genetics	Pricing
N	number of genes in a genotype	number of elements in a price (e.g. cost, markup, quality; c, m, q)
K	number of epistatic interactions between genes	interactions btw. pricing elements ($c \leftrightarrow m, m \leftrightarrow q$)
A	number of alleles for a gene	e.g. pricing states (*at-above-below* market)
C	degree of coupling between genotypes	price coupling between different markets (location, use)

Hypothetical Pricescape: N=3, K=2 with Assigned Fitness

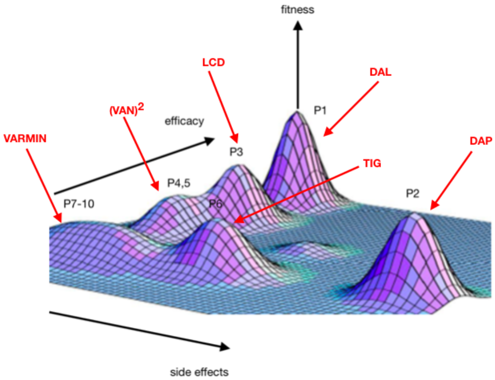


- Price dimensions: cost, markup, quality; c, m, q
- $N = 3$; scoring $\begin{cases} c & 0 : am, 1 : bm \\ m & 1 : am, 0 : bm \\ q & 1 : am, 0 : bm \end{cases}$
- K=2: interactions btw. pricing elements $(c \leftrightarrow m, m \leftrightarrow q)$

Real World Pricescape: Constructing an Antibiotics Commercial Landscape

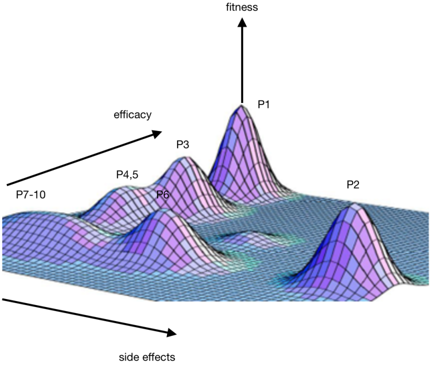
- Antibiotics Competition Space:
  span (Efficacy, Side Effects, Economic Fitness)
- Efficacy:
  Drug effectiveness, e.g. by UK-NICE or AAUSES scores
- Side Effects:
  Drug SE profile (GI, fungi, skin), e.g. by UK-NICE score
- Economic Fitness:
  - profitability: margin $m = p - c$ (*producer* view: price, cost)
  - value e.g. $v = u(x)$ (*consumer* view: perceived utility)

First Look at a Real-World Antibiotics Fitness Landscape



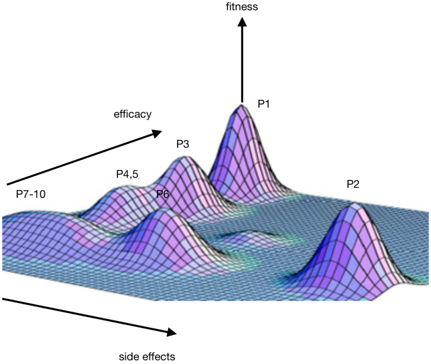
- Data from antibiotics use meta-analysis in acute skin infections; 7816 pts., 19 stds. (adapted fm. Guest et. al, 2017)
- DAL: dalbavancin, DAP: daptomycin, LCD: linecolid, TIG: tigecycline, VAN: vancomycin, VARMIN: var. minor antiotics

Generalization I: Static Analysis of a Pricescape



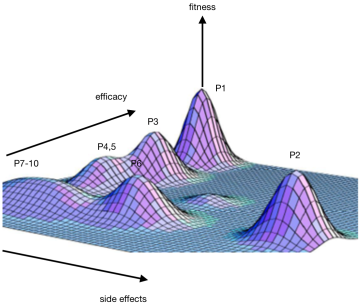
- P1: $f_1(m_1, v_1)$; m_1, v_1: max; *single (unique) global maximum*
- P2, P3, P6: *multiple (unique) local maxima*
- P4,5; P7-10: *multiple (mixed) local maxima*
- CW: definition of clusters, neighbourhood, distance $\rightarrow$ metric

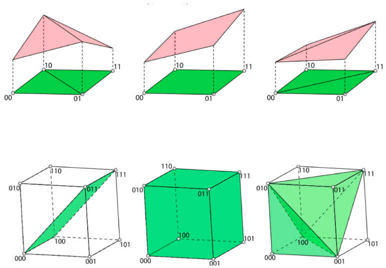
Generalization II: Dynamic Analysis of a Pricescape



- P1: primary; *price stable*
- P2, P3, P6: secondaries; *prices metastable*, pos. epistasis ?
- P4,5; P7-10: mixed; *prices unstable, disruptive*, neg. epistasis ?
- CW: modeling pos./neg.epistasis, neighbourhood distances

Pricescapes: Testing for Epistasis



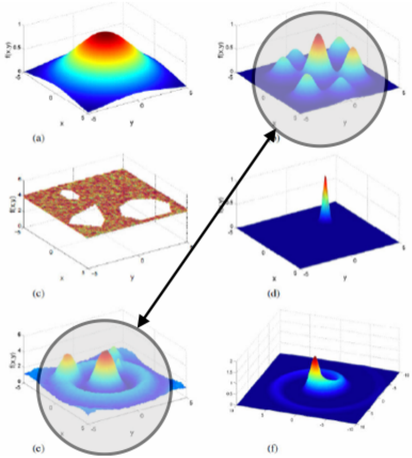


- (pictures from Gould et al., 2019): right panel shows two (top) and three agent interaction (bottom)
- P1, P3: two-way interaction coordinate suggests neg. epistasis $u_{P_1P_3} = F_{00} + F_{P_1P_3} - F_{0P_3} - F_{P_10} < 0$ (F = fitness); top left
- seems intuitively right (*'prices repel'*); more data required

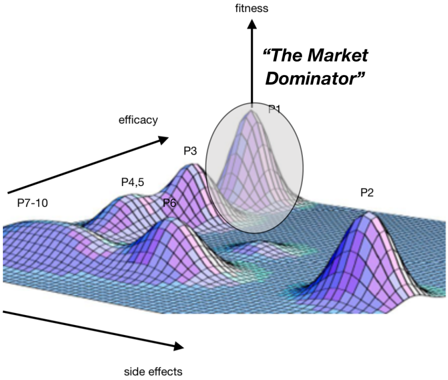
Pricescapes: Topologies and Pathways

Topological features and consequences for evolutionary pathways: How likely are the paths?

a) Single smooth peak (Mt. Fuji) → Evolutionary hill climbing
→ b) Rugged landscapes with multiple peaks → Basins of attraction and valley crossing
c) Holey landscapes → Finding ridged between optima
d) Neutral landscape with needle-in-the-haystack → Neutral drift, jumps in fitness and unguided search
→ e) Barrier landscape → Conditional valley crossing
f) Detour landscape (long-path problem) → Small paths

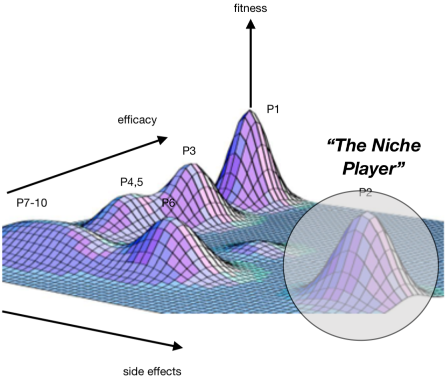


Pricescapes: Business Interpretation I - *Market Domination*



- P1: quasi-monopoly; leader, *price "at rest"*
- P2, P3, P6: (secondary) monopolies; followers, *prices "at rest"*
- P4,5; P7-10: (mixed); followers; *prices unstable, disruptive*

Pricescapes: Business Interpretation II - *Niche Player*



- P1: quasi-monopoly; leader, *price "at rest"*
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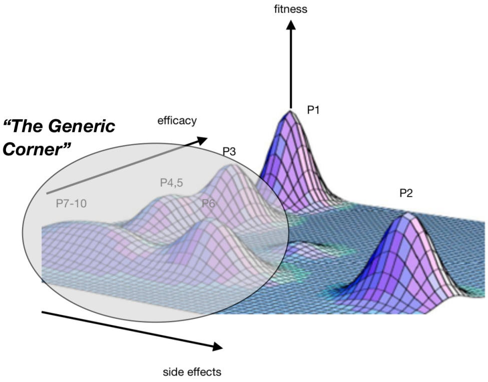
- P4,5; P7-10: (mixed); followers; *prices unstable, disruptive*

Pricescapes: Business Interpretation III - *Generic Corner*



- P1: quasi-monopoly; leader, *price "at rest"*
- P2, P3, P6: (secondary) monopolies; followers, *prices "at rest"*
- P4,5; P7-10: (mixed); followers; *prices unstable, disruptive*

Stochastic Mathematics & BSDEs:
From Data Description to Outcome Simulation

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Historical Starting Point IV: Backward Stochastic Differential Equations, Gobet 2003

- Backward Stochastic Differential Equation (BSDE) are important tools in mathematical finance. In a complete market, a contingent claim with payoff $\Phi(\mathbb{S})$, Y is the replicating portfolio value, Z is related to the hedging strategy.

- Numerical methods exist for solving decoupled forward-backward stochastic differential eqns. (FBSDE; Gobet, 2003).

$$S_t = S_0 + \underbrace{\int_0^t b\left(s, S_s\right) ds}_{\text{Riemann}} + \underbrace{\int_0^t \sigma\left(s, S_s\right) dW_s}_{\text{Ito}}$$

$$Y_t = \Phi(\mathbf{S}) + \int_t^T f\left(s, S_s, Y_s, Z_s\right) ds + \int_t^T Z_s dW_s$$

where $\mathbf{S} = (S_t : 0 \leq t \leq T)$ is the forward component and $\mathbf{Y} = (Y_t : 0 \leq t \leq T)$ is the backward one. Equations are solved in $\mathbf{S}, \mathbf{Y}$ and $\mathbf{Z}$ (Z: hedging strategy).

Numerical Solutions to BSDEs

- The authors propose a numerical method to find an approximation of the unique solution $(\mathbf{S}, \mathbf{Y}, \mathbf{Z})$ to the above equations
- The method is based on a Monte Carlo method involving a Voronoi partition. The authors also use a iterative method based on the Picard fix point theorem and a regression on function bases.
- The authors design a new algorithm for the numerical resolution of BSDEs. At each discretization time, it combines a finite number of Picard iterations and regressions on function bases.
- These regressions are evaluated rapidly with only one set of simulated paths.

Summary & Conclusions

- The evolution of biological, technological, social, *and* economical entities can be studied by a toolbox coming from differential equations, systems research, algebra, geometry and topology.
- The simplest models of rivalry correspond to systems of ordinary second order differential equations, widely used to describe numerous natural scientific objects (ODEs)
- More complex models rely on semilinear parabolic PDEs, a generalization of the Feynman-Kac type (BSDEs)
- Collaborative-competitive situations are modelled as fitness landscapes: evolutionary optimization methods using scalar valued fitness functions, s.a. potential functions in physics

Mathematics of Rivalry: Add-On Materials

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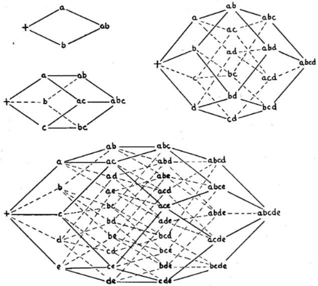
Selected References

- *High-dimensional microbiome interactions shape host fitness*. A. Gould, V. Zhang, L. Lamberti, E. Jones, B. Obadia, N. Korasidis, A. Gavryushkin, J. Carlson, N. Beerenwinkel, W. Ludington. Proc. Nat. Acad. Sci US, 2018, 10+83 pages
- *Pricing general insurance in a reactive and competitive Market*. P. Emms, J. Comp. Appl. Math., 2011, 19 pages
- *On a three-level competitive pricing problem with uniform and Mill pricing strategies*. A. Gubareva, A. Panin, A. Manganelli, A. Plyasunov, L. Som. J. Appl. Industr. Math., 2019, 23 pages
- *Comparative efficacy and safety of antibiotics to treat acute bacterial skin infections: Results of a network meta-analysis*. J. Guest, J. Esteban, A. Manganelli, A. Novelli, G. Rizzardini, M. Serra. PLoS ONE, 2017, 22+39 pages

Fitness Landscape Math I: Wright's Allellomorphs

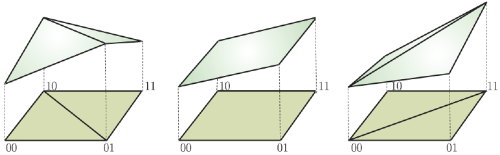
- Consider the set
  $\Sigma^k = \{w = (s_1, s_2, \ldots s_k) \,|s_i \in \Sigma, 1 \leq i \leq k\}$
  of words of length k in the alphabet
  $$\Sigma = \{n_1, n_2, \ldots, n_\ell\}$$

- A fitness landscape is a function
  $h : \Sigma^k \to \mathbb{R}$
  $$w \mapsto h(w)$$



Fitness Landscape Math II: Epistasis Introduction

- Epistasis (allelic interactions) in fitness landscapes



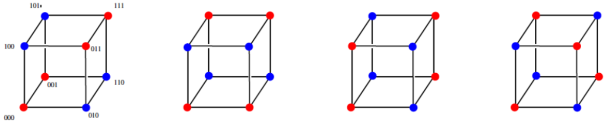
$$\epsilon(00,01,10,11) = h(11) + h(00) - h(10) - h(01)$$

$$\epsilon(00,01,10,11) \begin{cases} =0 \text{ no epistasis} \\ >0 \text{ positive epistasis} \\ <0 \text{ negative epistasis} \end{cases}$$

- Question:
  How to generalize this description to higher dimensions ?

Fitness Landscape Math III: Epistasis via Fourier Transform

- Interaction coordinates and Fourier transforms



- For $V = \{0,1\}^n$ and $w \in V$ with at least two coordinates equal to one, consider:

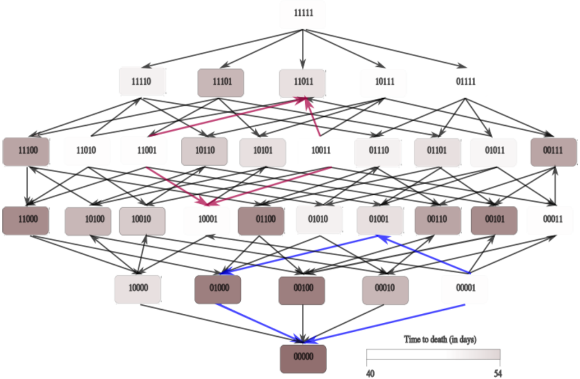
$$\sum_{v \in V} (-1)^{\langle v,w \rangle} h(v)$$

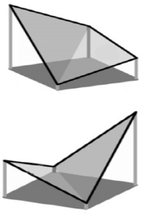
- For example, for $V = \{0,1\}^3$ and $w = 111$:

$$(h(000)+h(011)+h(101)+h(110))-(h(001)+h(010)+h(100)+h(111$$

Fitness Landscape Math IV: Fitness Graph Approximation

- Approximation via fitness graphs



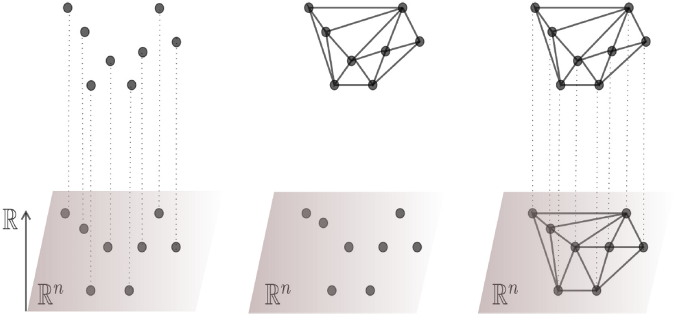


$h(00)+h(11)-h(01)-h(10)>0$

$\Longrightarrow$ Positive epistasis

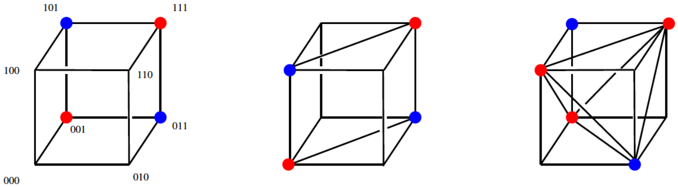
Fitness Landscape Math V: Shape Analysis

- Shapes of fitness landscvapes reveal epistasis



Fitness Landscape Math VI: Shapes via Circuits

- Shapes of fitness landscapes via circuits

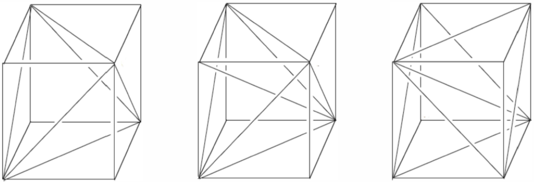


- For $V = \{0,1\}^3$, up to symmetries, consider the following circuit interactions:

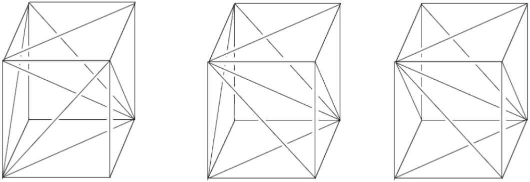
$$
\begin{aligned}
h_f &= h(111) + h(001) - h(101) - h(011) \\
h_p &= h(111) + h(000) - h(100) - h(011) \\
h_b &= h(111) + h(100) + h(001) - h(010) - 2h(101)
\end{aligned}
$$

Fitness Landscape Math VII: Triangulations

- Characterization via triangulation

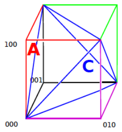


n	1	2	3	4	5	6	7
Regular triangulations of $[0,1]^n$	1	2	74	87959448	-	-	-
Circuits in $[0,1]^n$	1	1	20	1348	353616	-	-



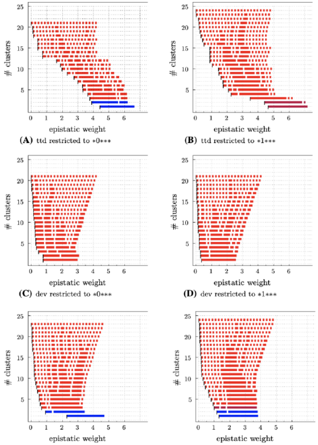
Fitness Landscape Math VIII: Cluster Partition Analysis

$$000 \mapsto 53.25\,;\quad 100 \mapsto 46.65\,;\quad 001 \mapsto 43.16\,;\quad 101 \mapsto 43.48\,;$$
$$010 \mapsto 48.3\,;\quad 110 \mapsto 47.79\,;\quad 011 \mapsto 43.53\,;\quad 111 \mapsto 40.71\,.$$



Epistatic weight:

$$e_h(\mathbf{A},\mathbf{C}) = \left|\det\begin{pmatrix} 1 & 0 & 0 & 0 & 53.25 \\ 1 & 1 & 0 & 0 & 46.65 \\ 1 & 1 & 1 & 0 & 47.79 \\ 1 & 1 & 0 & 1 & 43.48 \\ 1 & 0 & 1 & 1 & 43.53 \end{pmatrix}\right| \cdot \frac{\sqrt{3}}{2}$$
$$= \left| h(000) + h(110) + h(101) - h(011) - 2h(100) \right| \cdot \frac{\sqrt{3}}{2}$$
$$= 6.660\ .$$
$$= \det(E_h(A,C)) \cdot \frac{\mathrm{nvol}(A \cap C)}{\mathrm{nvol}\, A \cdot \mathrm{nvol}\, C}$$



(A) ttd restricted to *0***
(B) ttd restricted to *1***
(C) dev restricted to *0***
(D) dev restricted to *1***
(E) fec restricted to *0***
(F) fec restricted to *1***.

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